Nuclear Direct Reactions to Continuum 2

– How to get Nuclear Structure Information –

Munetake ICHIMURA (RNC)

III PWBA I

1. PWBA

As the first stage 1

Let's consider reactions in **PWBA**
(Plane Wave Born Approximation)

- This is an ideal clean impulse model.
- This may not be realistic, but tells us fundamental structure of the reactions.
- We can learn what information can be obtained from the reaction considered.
- This is very important to design (or analyse) experiments.

Let's consider a final two-body reaction

$$a + A \longrightarrow b + B$$

usually written as

$$A(a, b)B$$

1.1 Born Approximation

Assume that the interaction works only once

$$T_{fi} = \langle \phi_f \Phi_b \Phi_B | V | \Phi_A \Phi_a \phi_i \rangle$$

Φ_A : wave function of the particle A
Φ_B : wave function of the particle B
Φ_a : wave function of the particle a
Φ_b : wave function of the particle b
ϕ_i : w. f. of the relative motion between a and A
ϕ_f : w. f. of the relative motion between b and B
V : interaction between a and A or b and B

1.2 Plane Wave Approximation

Assume that the relative motions are described by the plane wave

$$\phi_i = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i \cdot \boldsymbol{R}_i}, \qquad \phi_f = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f \cdot \boldsymbol{R}_f}$$

$\boldsymbol{R}_i$: Relative coordinate between a and A
$\boldsymbol{R}_f$: Relative coordinate between b and B
$\boldsymbol{k}_i$: Momentum of the relative motion in the initial channel
$\boldsymbol{k}_f$: Momentum of the relative motion in the final channel

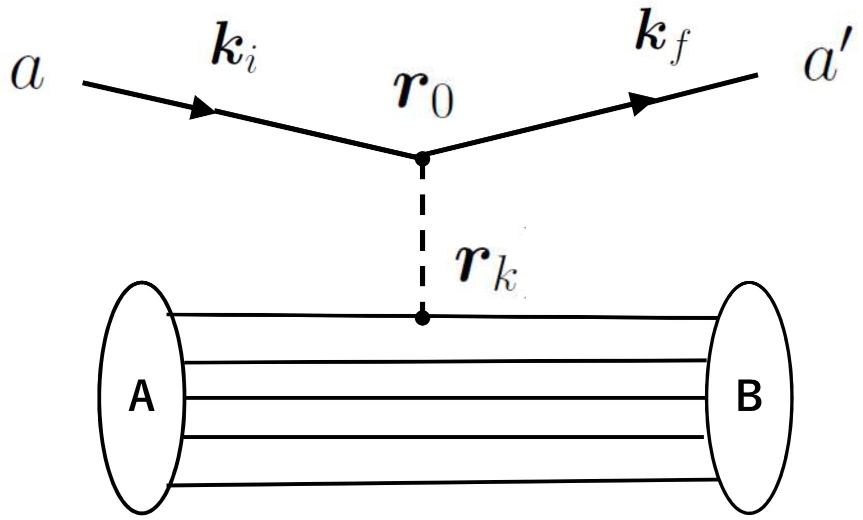
1.3 PWBA

Plane wave approx. + Born approx.

$$T_{fi} = \langle \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f \cdot \boldsymbol{R}_f} \Phi_b \Phi_B | V | \Phi_A \Phi_a \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i \cdot \boldsymbol{R}_i} \rangle$$

2. Fundamental Examples

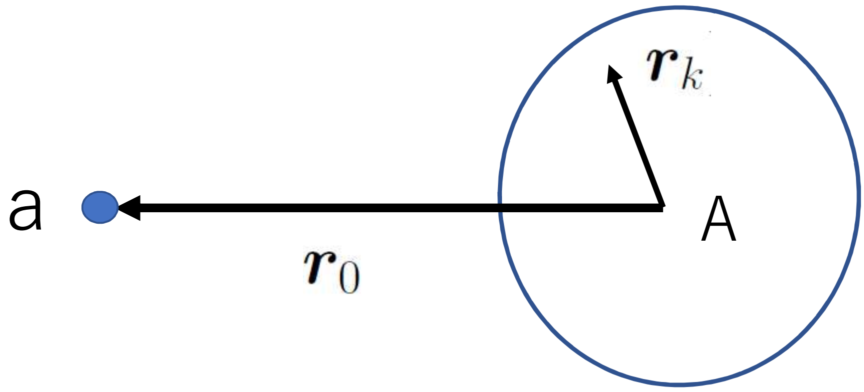
2.1. Simplest (fundamental) case



Consider a reaction

$$A(a, a')B$$

$a,\ a'$: structureless point particle



- **Interaction**

Sum of the two-body interaction

$$V = \sum_{k \in A} V(\boldsymbol{r}_0 - \boldsymbol{r}_k)$$

Its Fourier transform

$$V(\boldsymbol{r}_0 - \boldsymbol{r}_k) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \tilde{V}(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}_0 - \boldsymbol{r}_k)}$$

- Wave functions of the relative motion

$$\phi_i = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}_0}, \qquad \phi_f = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_0}$$

● Calculation of the T-matrix

$$
\begin{aligned}
T_{fi} &= \langle \phi_f \Phi_B | V | \Phi_A \phi_i \rangle \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \tilde{V}(\boldsymbol{p}) \\
&\times \int d^3\boldsymbol{r}_0\; \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_0}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}_0}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}_0} \\
&\times \langle \Phi_B | \sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}_k} | \Phi_A \rangle \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \tilde{V}(\boldsymbol{p})\,(2\pi)^3 \delta(\boldsymbol{k}_i + \boldsymbol{p} - \boldsymbol{k}_f) \\
&\times \langle \Phi_B | \sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}_k)} | \Phi_A \rangle \\
&= \tilde{V}(\boldsymbol{q}^*)\; \langle \Phi_B | \sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^*\cdot\boldsymbol{r}_k} | \Phi_A \rangle
\end{aligned}
$$

with **Momentum Transfer**

$$\boldsymbol{q}^* = \boldsymbol{k}_f - \boldsymbol{k}_i$$

Density operator

We define the density operator

$$\rho(\boldsymbol{r}) = \sum_{k=1}^{A} \delta(\boldsymbol{r} - \boldsymbol{r}_k) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \tilde{\rho}(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}$$

then

$$\tilde{\rho}(\boldsymbol{p}) = \int d^3\boldsymbol{r}\, \rho(\boldsymbol{r}) \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} = \sum_{k=1}^{A} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}_k}$$

Transition form factor

We define the transition form factor

$$\begin{aligned} F_{BA}(\boldsymbol{q}^*) &\equiv \langle \Phi_B | \sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^*\cdot\boldsymbol{r}_k} | \Phi_A \rangle \\ &= \langle \Phi_B | \tilde{\rho}(\boldsymbol{q}^*) | \Phi_A \rangle \end{aligned}$$

T-matrix

$$T_{fi} = \tilde{V}(\boldsymbol{q}^*)\ F_{BA}(\boldsymbol{q}^*)$$

● Differential cross section

$$\frac{d\sigma}{d\Omega} = K \, |V(\boldsymbol{q}^*)|^2 \; |F_{A^*A}(\boldsymbol{q}^*)|^2$$

reaction part　　structure part

○ The reaction part and the structure part are factorized !

○ Determined only by the momentum transfer $\boldsymbol{q}^*$ except for the kinematical factor

$$K = \frac{\mu_i \mu_f}{(2\pi)^2} \frac{k_f}{k_i}$$

[Comment 1]

Note the relation $q^* \Leftrightarrow \theta$

$$q^* = |\boldsymbol{q}^*| = \sqrt{k_i^2 + k_f^2 - 2k_i k_f \cos\theta}$$

We can easily guess the angular distribution.

[Comment 2]

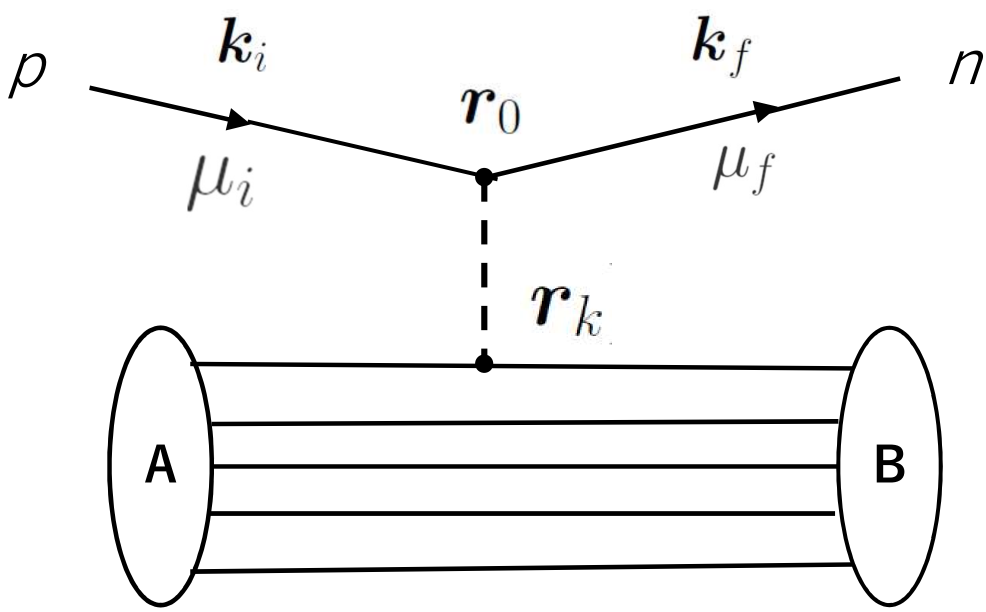
Note the restriction !

$$\sum_{k=1}^{A} \boldsymbol{r}_k = 0$$

Exactly speaking $\rho(\boldsymbol{r})$ is not a one-body operator.

Forget for a while ! We will touch later

2.2 Cases with spin and isospin



Consider a reaction

$$A(p,n)B$$

μ_i, μ_f : z-component of the nucleon spin

2.2.1 Isospin operators

(Pauli) isospin matrices of the nucleon $\boldsymbol{\tau}$

$$\boldsymbol{\tau} \longleftarrow \boldsymbol{\sigma}$$

Isospin operators

$$\boldsymbol{t} = \frac{1}{2}\boldsymbol{\tau}$$

Isospin raising and lowering operators

$$t^+ = t_x + \mathrm{i}t_y = \frac{1}{2}(\tau_x + \mathrm{i}\tau_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$t^- = t_x - \mathrm{i}t_y = \frac{1}{2}(\tau_x - \mathrm{i}\tau_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Convention in nuclear physics

$$\tau_z|n\rangle = |n\rangle, \quad \tau_z|p\rangle = -|p\rangle$$

$$|n\rangle = t^+|p\rangle, \quad |p\rangle = t^-|n\rangle$$

2.2.2 Case 1

- **Interaction**

$$
\begin{aligned}
V &= \sum_k (\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)(\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_k) V_{\tau\sigma}(\boldsymbol{r}_0 - \boldsymbol{r}_k) \\
&= \sum_k (\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)(\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_k) \\
&\times \int \tilde{V}_{\tau\sigma}(\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}_0 - \boldsymbol{r}_k)}\, \frac{d^3\boldsymbol{p}}{(2\pi)^3}
\end{aligned}
$$

- **T-matrix**

$$
\begin{aligned}
T_{fi} = \tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)\ &\langle \mu_f, n | \boldsymbol{\tau}_0 \boldsymbol{\sigma}_0 | \mu_i, p \rangle \\
&\cdot\ \langle \Phi_B | \sum_k \boldsymbol{\tau}_k \boldsymbol{\sigma}_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^* \cdot \boldsymbol{r}_k} | \Phi_A \rangle
\end{aligned}
$$

Calculation of the isospin part

$$\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k = t_0^+ t_k^- + t_0^- t_k^+ + \tau_{z,0}\tau_{z,k}$$

thus

$$\langle n|\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k|p\rangle = t_k^-$$

The T-matrix is now written as

$$\begin{aligned} T_{fi} &= \tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)\ \langle \mu_f|\boldsymbol{\sigma}_0|\mu_i\rangle \\ &\quad \cdot\ \langle \Phi_B|\sum_k t_k^- \boldsymbol{\sigma}_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle \\ &= \tilde{V}_{\tau\sigma}(\boldsymbol{q}^*) \sum_a [\sigma_{a,0}]_{\mu_f,\mu_i} F_{BA}^{(-,a)}(\boldsymbol{q}^*) \end{aligned}$$

where $(a = x, y, z)$ and

$$F_{BA}^{(-,a)}(\boldsymbol{q}) = \langle \Phi_B|\sum_k t_k^- \sigma_{a,k} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}|\Phi_A\rangle$$

Calculation of the differential cross section

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= K \ |\tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)|^2 \\
&\times \sum_{\mu_f} \frac{1}{2} \sum_{\mu_i} \sum_{ab} \langle \mu_f | \sigma_{a,0} | \mu_i \rangle^* \langle \mu_f | \sigma_{b,0} | \mu_i \rangle \\
&\times \left\langle \Phi_B \right| \sum_k t_k^- \sigma_{a,k} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^* \cdot \boldsymbol{r}_k} \left| \Phi_A \right\rangle^* \\
&\times \left\langle \Phi_B \right| \sum_k t_k^- \sigma_{b,k} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^* \cdot \boldsymbol{r}_k} \left| \Phi_A \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= K \ |\tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)|^2 \frac{1}{2} \sum_{ab} \mathrm{Tr} \left[\sigma_a \sigma_b \right] \\
&\times \left\langle \Phi_B \right| \sum_k t_k^- \sigma_{a,k} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^* \cdot \boldsymbol{r}_k} \left| \Phi_A \right\rangle^* \\
&\times \left\langle \Phi_B \right| \sum_k t_k^- \sigma_{b,k} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^* \cdot \boldsymbol{r}_k} \left| \Phi_A \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= K \ |\tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)|^2 \\
&\times \sum_a \left| \left\langle \Phi_B \right| \sum_k t_k^- \sigma_{a,k} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^* \cdot \boldsymbol{r}_k} \left| \Phi_A \right\rangle \right|^2
\end{aligned}
$$

Summing up the calculation, we get

- **T-matrix**

$$T_{fi} = \tilde{V}_{\tau\sigma}(\boldsymbol{q}^*) \ \sum_a [\sigma_a]_{\mu_f,\mu_i} F_{BA}^{(-,a)}(\boldsymbol{q}^*)$$

- **Isovector (IV) spin-vector (SV) transition form factor**

$$F_{BA}^{(\pm,a)}(\boldsymbol{q}^*) = \langle \Phi_B | \tilde{\rho}_a^{(\pm)}(\boldsymbol{q}^*) | \Phi_A \rangle$$

- **IV-SV transition density**

$$\tilde{\rho}_a^{(\pm)}(\boldsymbol{q}) = \sum_k t_k^{\pm} \sigma_{a,k} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}$$

● Differential cross section

$$\frac{d\sigma}{d\Omega} = K\ |\tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)|^2\ \sum_a\ |F_{BA}^{(-,a)}(\boldsymbol{q}^*)|^2$$

The reaction part and the structure part are factorized again !

2.2.3 Case 2

- **Interaction**

$$V = \sum_k (\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)\ V_\tau(\boldsymbol{r}_0 - \boldsymbol{r}_k) + \sum_k (\boldsymbol{\tau}_0 \cdot \boldsymbol{\tau}_k)(\boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_k)\ V_{\tau\sigma}(\boldsymbol{r}_0 - \boldsymbol{r}_k)$$

- **T-matrix**

$$T_{fi} = \tilde{V}_\tau(\boldsymbol{q}^*)\ \delta_{\mu_f,\mu_i} F_{BA}^{(-)}(\boldsymbol{q}^*) + \tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)\ \sum_a [\sigma_a]_{\mu_f,\mu_i} F_{BA}^{(-,a)}(\boldsymbol{q}^*)$$

- **Isovector (IV) spin-scalar (SS) transition form factor**

$$F_{BA}^{(\pm)}(\boldsymbol{q}) = \langle \Phi_B | \tilde{\rho}^{\pm}(\boldsymbol{q}) | \Phi_A \rangle$$

- **IV-SS transition density**

$$\tilde{\rho}^{\pm}(\boldsymbol{q}) = \sum_k t_k^{\pm} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k}$$

How about the interference between the IV-SS and IV-SV parts ?

$$\text{Interference term} \propto \sum_{\mu_f} \frac{1}{2} \sum_{\mu_i} \delta_{\mu_f,\mu_i} \left[\sigma_a\right]_{\mu_f,\mu_i} \cdots$$
$$= \frac{1}{2}\text{Tr}[\sigma_a] \cdots = 0$$

The sum of the spin z-components $\frac{1}{2}\sum_{\mu_f}\sum_{\mu_i}$ is crucial to cut the interference term.

This is a characteristic of PWBA !

- **Differential cross section**

$$\frac{d\sigma}{d\Omega} = \frac{\mu_i \mu_f}{(2\pi)^2} \frac{k_f}{k_i} \left\{ |\tilde{V}_\tau(\boldsymbol{q}^*)|^2 \, |F_{BA}^{(-)}(\boldsymbol{q}^*)|^2 + |\tilde{V}_{\tau\sigma}(\boldsymbol{q}^*)|^2 \sum_a |F_{BA}^{(-),a}(\boldsymbol{q}^*)|^2 \right\}$$

2.2.4. Special case ($\boldsymbol{q}^* = 0$)

At $\boldsymbol{q}^* = 0$, the structure parts become

(1) Transition strength
to the Isobaric Analogue State (IAS)

$$
\begin{aligned}
B(\mathrm{IAS}^{\pm} : A \to B) &= |\langle \Phi_B | \sum_k t_k^{\pm} | \Phi_A \rangle|^2 \\
&= |F_{BA}^{(\pm)}(\boldsymbol{q}^* = 0)|^2
\end{aligned}
$$

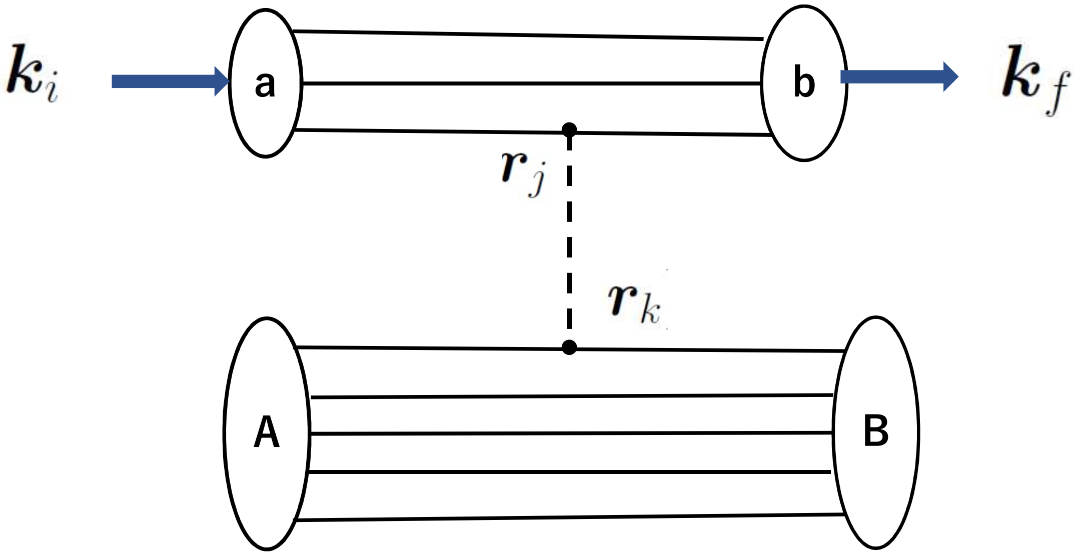
(2) Gamow-Teller (GT) transition strength

$$
\begin{aligned}
B(\mathrm{GT}^{\pm} : A \to B) &= \sum_a |\langle \Phi_B | \sum_k t_k^{-} \sigma_{a,k} | \Phi_A \rangle|^2 \\
&= \sum_a \; |F_{BA}^{(\pm,a)}(\boldsymbol{q}^* = 0)|^2
\end{aligned}
$$

- These are the key examples to extract structure information from reactions.

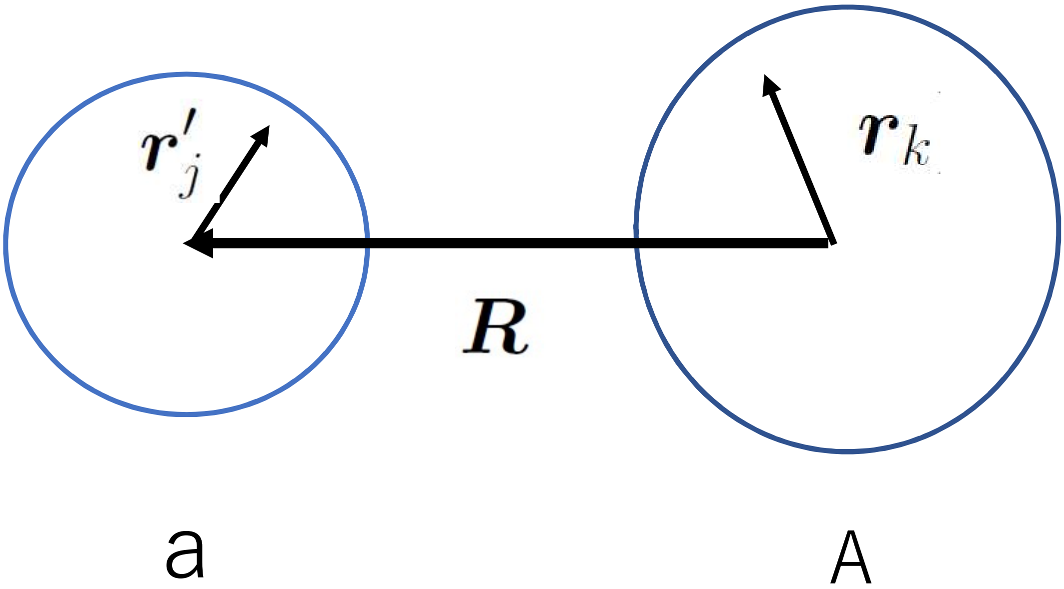
- But can we get the form factors at $\boldsymbol{q}^* = 0$?
  Unfortunately No ! in general.
  We need tricks.

2.3 Reaction of composite particles



- **Interaction**

$$V = \sum_{j\in a}\sum_{k\in A} V(\boldsymbol{r}_j - \boldsymbol{r}_k)$$
$$= \sum_{j\in a}\sum_{k\in A}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\tilde{V}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}'_j+\boldsymbol{R}-\boldsymbol{r}_k)}$$



Take coordinates

$$\boldsymbol{r}_j = \boldsymbol{r}'_j + \boldsymbol{R}$$

Plane waves of the relative motion

$$\phi_i = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{R}}, \qquad \phi_f = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{R}}$$

Carry out the integration over $\boldsymbol{R}$, we get

$$T_{fi} = \tilde{V}(\boldsymbol{q}^*) \, \langle\Phi_b| \sum_{j\in a} \mathrm{e}^{\mathrm{i}\boldsymbol{q}^*\cdot\boldsymbol{r}'_j}|\Phi_a\rangle$$
$$\times \qquad \langle\Phi_B| \sum_{k\in A} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^*\cdot\boldsymbol{r}_k}|\Phi_A\rangle$$

We reach the formulas

- **T-matrix**

$$T_{fi} = \tilde{V}(\boldsymbol{q}^*)\ F_{ba}(-\boldsymbol{q}^*)\ F_{BA}(\boldsymbol{q}^*)$$

- **Transition form factors**

$$F_{ba}(\boldsymbol{q}) = \langle \Phi_b | \sum_{j \in a} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}'_j} | \Phi_a \rangle$$

$$F_{BA}(\boldsymbol{q}) = \langle \Phi_B | \sum_{k \in A} \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_k} | \Phi_A \rangle$$

- **Ddifferential cross section**

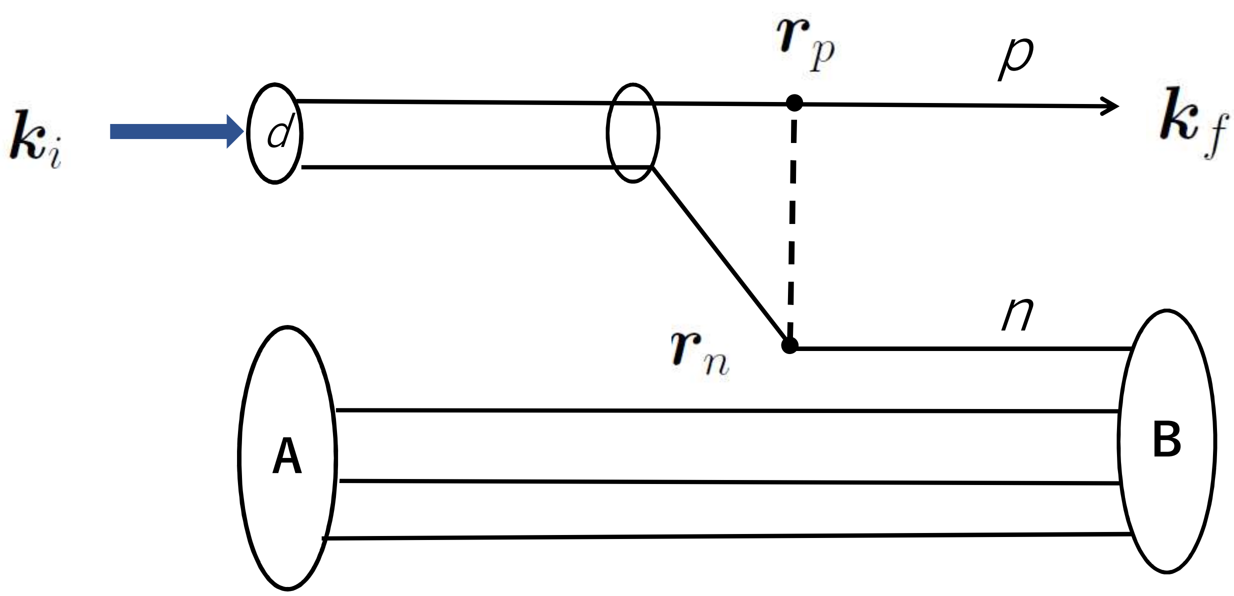
$$\frac{d\sigma}{d\Omega} = \frac{\mu_i \mu_f}{(2\pi)^2} \frac{k_f}{k_i}\ |\tilde{V}(\boldsymbol{q}^*)|^2 \qquad \text{(reaction part)}$$

$$\times\ |F_{ba}(-\boldsymbol{q}^*)|^2\ |F_{BA}(\boldsymbol{q}^*)|^2 \qquad \text{(structure part)}$$

2.4 Rearrangement collision

Consider a reaction

$$A(d,p)B$$



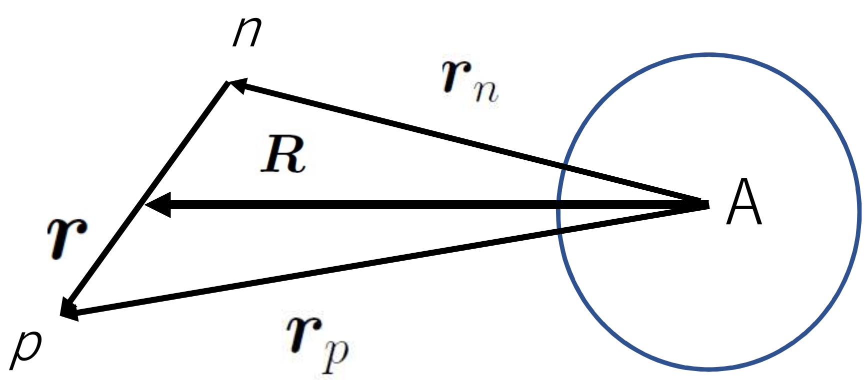
Assume

- A is inert core.
- Neglect the spins
- Interaction

$$V = V_{pn}(\boldsymbol{r}_p - \boldsymbol{r}_n)$$

Use the coordinates

$$\boldsymbol{r} = \boldsymbol{r}_p - \boldsymbol{r}_n, \qquad \boldsymbol{R} = \frac{\boldsymbol{r}_p + \boldsymbol{r}_n}{2}$$



Wave functions

$$\phi_i = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i \cdot \boldsymbol{R}}, \qquad \phi_f = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f \cdot \boldsymbol{r}_p}$$

$$\Phi_a = \phi_d(\boldsymbol{r}), \qquad \Phi_B = \Phi_A \psi_n(\boldsymbol{r}_n)$$

T-matrix

$$T_{fi} = \int d^3\boldsymbol{r}_p \int d^3\boldsymbol{r}_n \left\{ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_p} \ \psi_n^*(\boldsymbol{r}_n) \right\} \\ \times V_{pn}(\boldsymbol{r}) \left\{ \phi_d(\boldsymbol{r}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{R}} \right\}$$

Fourier transform $V_{pn}(\boldsymbol{r})\phi_d(\boldsymbol{r})$

$$V_{pn}(\boldsymbol{r})\phi_d(\boldsymbol{r}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \ D(\boldsymbol{p}) \ \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}$$

The T-matrix becomes

$$\begin{aligned} T_{fi} &= \int d^3\boldsymbol{r}_p \int d^3\boldsymbol{r}_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} D(\boldsymbol{p}) \\ &\times \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_p} \psi_n^*(\boldsymbol{r}_n) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{R}} \\ &= \int d^3\boldsymbol{r}_p \int d^3\boldsymbol{r}_n \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} D(\boldsymbol{p}) \\ &\times \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_p} \psi_n^*(\boldsymbol{r}_n) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}_p-\boldsymbol{r}_n)} \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot(\boldsymbol{r}_p+\boldsymbol{r}_n)/2} \\ &= D\left(\boldsymbol{k}_f - \frac{\boldsymbol{k}_i}{2}\right) \int d^3\boldsymbol{r}_n \psi_n^*(\boldsymbol{r}_n) \ \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_f-\boldsymbol{k}_i)\cdot\boldsymbol{r}_n} \\ &= D\left(\boldsymbol{k}_f - \frac{\boldsymbol{k}_i}{2}\right) \ \psi_n^*(-\boldsymbol{q}^*) \end{aligned}$$

● Differential cross section

$$\frac{d\sigma}{d\Omega} = K \left| D\left(\boldsymbol{k}_f - \frac{\boldsymbol{k}_i}{2}\right)\right|^2 |\psi_n(-\boldsymbol{q}^*)|^2$$

○ We may get information about the neutron wave function in B.

[Comment 1]

● Zero range approximation

Has been widely used for (d,p) reaction

$$V_{pn}(\boldsymbol{r})\phi_d(\boldsymbol{r}) = D_0\delta(\boldsymbol{r})$$

means

$$D(\boldsymbol{p}) = D_0$$

Then

$$\frac{d\sigma}{d\Omega} = K\ D_0^2\ |\psi_n(-\boldsymbol{q})|^2$$

[Just for fun]

Schrödinger equation for the deuteron

$$\left(-\frac{\boldsymbol{\nabla}_{\boldsymbol{r}}^2}{m_N}+V(\boldsymbol{r})\right)\phi_d(\boldsymbol{r})=-\epsilon_d\phi_d(\boldsymbol{r})$$

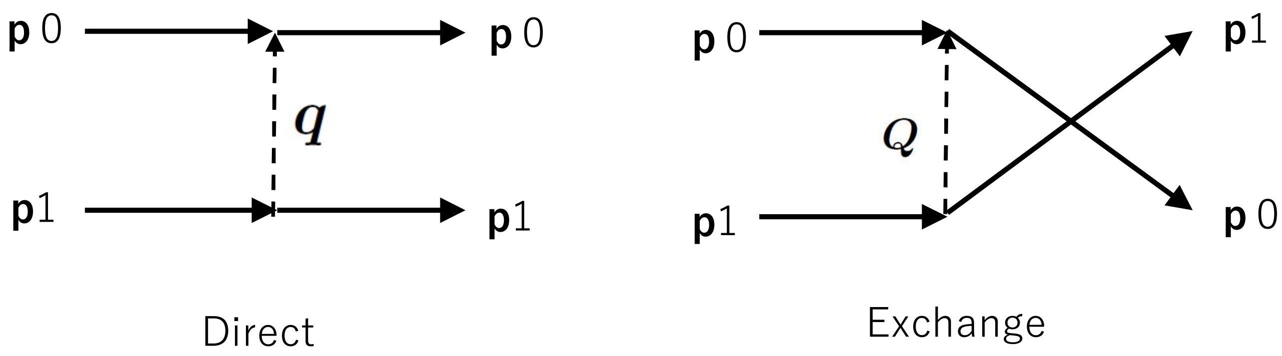
ϵ_d : Binding energy of the deuteron

Fourier transform of $V(\boldsymbol{r})\phi(\boldsymbol{r})$

$$\begin{aligned}D(\boldsymbol{p}) &= \int d^3\boldsymbol{r}V(\boldsymbol{r})\phi_d(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \\ &= \int d^3\boldsymbol{r}\left(\frac{\boldsymbol{\nabla}_{\boldsymbol{r}}^2}{m_N}-\epsilon_d\right))\phi_d(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \\ &= -\left(\frac{\boldsymbol{p}^2}{m_N}+\epsilon_d\right)\phi_d(\boldsymbol{p})\end{aligned}$$

2.5 Exchange processes

2.5.1. NN scattering



Consider a nucleon-nucleon (NN) scattering

$$p + p \to p + p$$

NN scattering t-matrix

$$t_{NN} = t_{NN}^{D} - t_{NN}^{E}$$

We cannot distinguish p_0 and p_1

Ignore spin and isospin for simplicity.
Just learn the essence.

(1) Direct process

$$\begin{aligned} t_{NN}^{D} &= \langle \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_0}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_1}|V(\boldsymbol{r}_0-\boldsymbol{r}_1)|\mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}_0}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}_1}\rangle \\ &= \tilde{V}(\boldsymbol{q}^*) \end{aligned}$$

(2) Exchange process

$$\begin{aligned} t_{NN}^{E} &= \langle \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_1}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_0}|V(\boldsymbol{r}_0-\boldsymbol{r}_1)|\mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}_0}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}_1}\rangle \\ &= \tilde{V}(\boldsymbol{Q}^*) \end{aligned}$$

with

$$\boldsymbol{Q}^* = -(\boldsymbol{k}_f+\boldsymbol{k}_i) = -(2\boldsymbol{k}_i+\boldsymbol{q})$$

● Pseudo-potential approximation

High energy forward scattering

$$q^* \ll 2k_i$$

We may use an approximation

$$t_{NN}^E = \tilde{V}(\boldsymbol{Q}^*) \approx \tilde{V}(-2\boldsymbol{k}_i)$$

$\tilde{V}(-2\boldsymbol{k}_i)$: a constant with respect to $\boldsymbol{q}^*$,
determined by the initial state

Now we can calculate the full t_{NN}
by only the direct term of the potential

$$V = V(\boldsymbol{r}_0 - \boldsymbol{r}_1) - V_0 \delta(\boldsymbol{r}_0 - \boldsymbol{r}_1)$$

where

$$V_0 = \tilde{V}(-2\boldsymbol{k}_i)$$

The 2nd term : **Pseudo-potential**

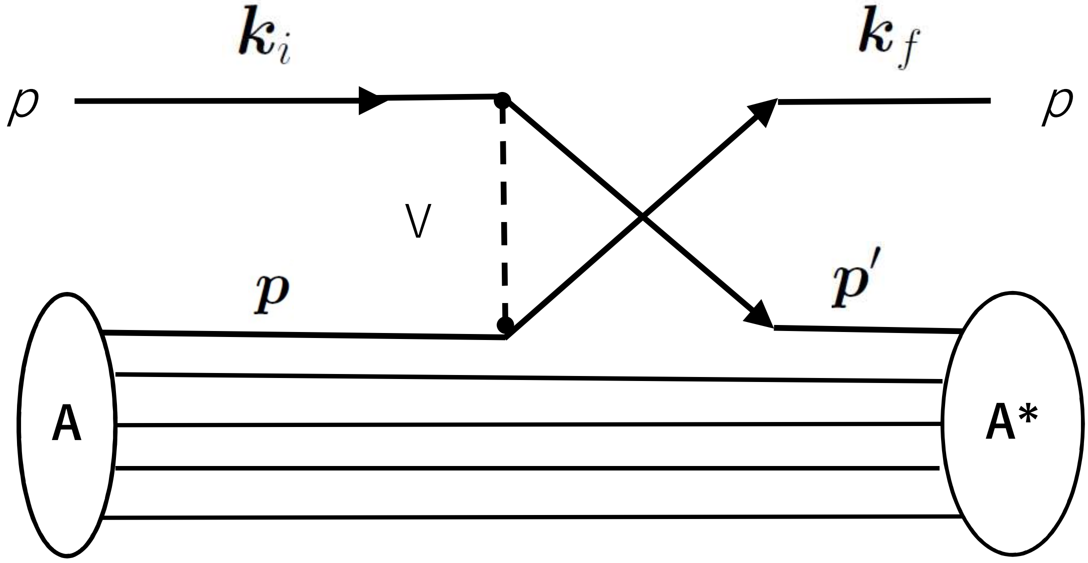
○ This prescription is very useful
to represent the exchange effects
by the direct processes
via a local potential !

○ In realistic cases, we must consider

- spins, isospins
- tensor forces
- velocity dependent forces

etc.

2.5.2. Nucleon-nucleus scattering (NA scattering)



Consider the exchange process in

$$A(p, p')A^*$$

Ignore spin and isospin.

- Initial state

$$|i\rangle = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}(\boldsymbol{p}|\Phi_A\rangle \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}_k}\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}_0}$$

- Final state

$$|f\rangle = \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3}(\boldsymbol{p}'|\Phi_{A^*}\rangle \mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{r}_0}\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_k}$$

- Interaction

$$V(\boldsymbol{r}_0 - \boldsymbol{r}_k) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\tilde{V}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_k)}$$

Use the momentum conservation at each vertex.

T-matrix for the exchange process is now written as

$$T^{\mathrm{E}}_{fi} = -\langle f|V(\boldsymbol{r}_0 - \boldsymbol{r}_k)|i\rangle$$
$$= -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\langle \phi^*_A|\boldsymbol{p} - \boldsymbol{q}^*)\tilde{V}(\boldsymbol{p} - \boldsymbol{q}^* - \boldsymbol{k}_i)(\boldsymbol{p}|\phi_A\rangle$$

Noting

$$p, p' \;\leq\; k_F \approx 1.4 \text{ fm}^{-1}$$

Momentum of 300MeV proton

$$k = \sqrt{E_p^2 - m_p^2} = 808 \text{ MeV} \approx 4.0 \text{ fm}^{-1}$$

We may take an approximation

$$\tilde{V}(\boldsymbol{p} - \boldsymbol{q}^* - \boldsymbol{k}_i) = \tilde{V}(\boldsymbol{p}' - \boldsymbol{k}_i) \approx \tilde{V}(-\boldsymbol{k}_i)$$

which is a constant for the given $\boldsymbol{k}_i$.

Now the T-matrix for the exchange process becomes

$$\begin{aligned} T_{fi}^{\mathrm{E}} &= -\tilde{V}(-\boldsymbol{k}_i) \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \langle \phi_A^* | \boldsymbol{p} - \boldsymbol{q}^*)(\boldsymbol{p} | \phi_A \rangle \\ &= -\tilde{V}(-\boldsymbol{k}_i) \langle \phi_{A^*} | \sum_k \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^* \cdot \boldsymbol{r}_k} | \phi_A \rangle \\ &= -\tilde{V}(-\boldsymbol{k}_i) F_{A^* A}(\boldsymbol{q}^*) \end{aligned}$$

Use the interaction with pseudo-potential

$$V = V(\boldsymbol{r}_0 - \boldsymbol{r}_1) - \tilde{V}(-\boldsymbol{k}_i)\delta(\boldsymbol{r}_0 - \boldsymbol{r}_1)$$

and calculate only the direct term.

We get the full T-matrix as

$$T_{fi} = \left[\tilde{V}(\boldsymbol{q}*) - \tilde{V}(-\boldsymbol{k}_i)\right] \; F_{A^*A}(\boldsymbol{q}^*)$$

In this approximation

$$T_{fi} \propto F_{A^*A}(\boldsymbol{q}^*)$$

Very useful !

[Comment]

- $\boldsymbol{k}_i$ in the previous subsection is the incident momentum in the cm frame of the NN system,
- $\boldsymbol{k}_i$ here is the incident momentum in the cm frame of the NA system

Note

$$2k_i^{\rm NN} = k_i^{\rm NA} \approx k_{i,\rm lab} \quad \text{for } m_A \to \infty$$

For a central potential

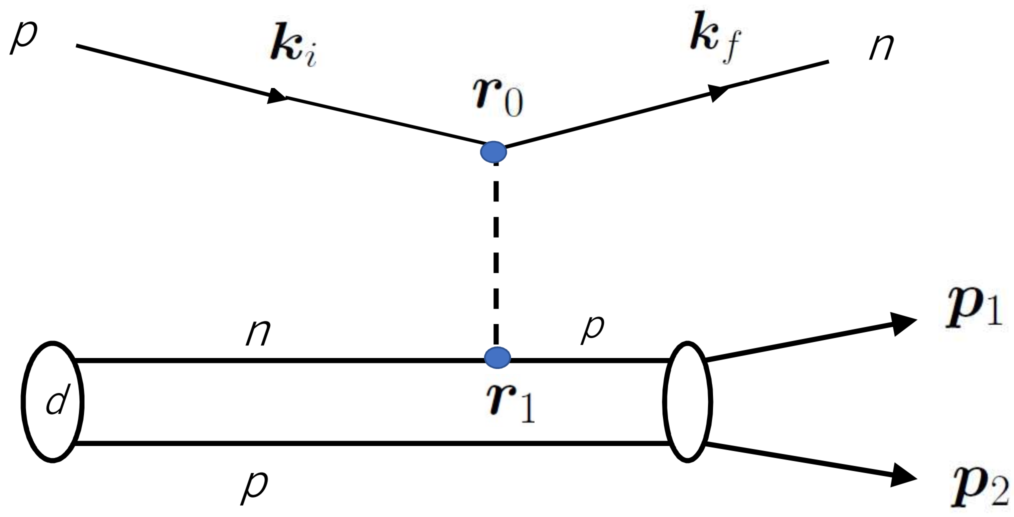
$$\tilde{V}(\boldsymbol{p}) = \tilde{V}(p)$$

Thus we can set

$$\begin{aligned} &\tilde{V}(-2\boldsymbol{k}_i^{NN})) = \tilde{V}(2k_i^{NN})) \\ \approx\ &\tilde{V}(-\boldsymbol{k}_i^{NA})) = \tilde{V}(k_i^{NA})) \\ \approx\ &\tilde{V}(k_{i,\rm lab})) \end{aligned}$$

IV PWBA II
– Reaction to Continuum

1. Simplest reaction



Consider the reaction

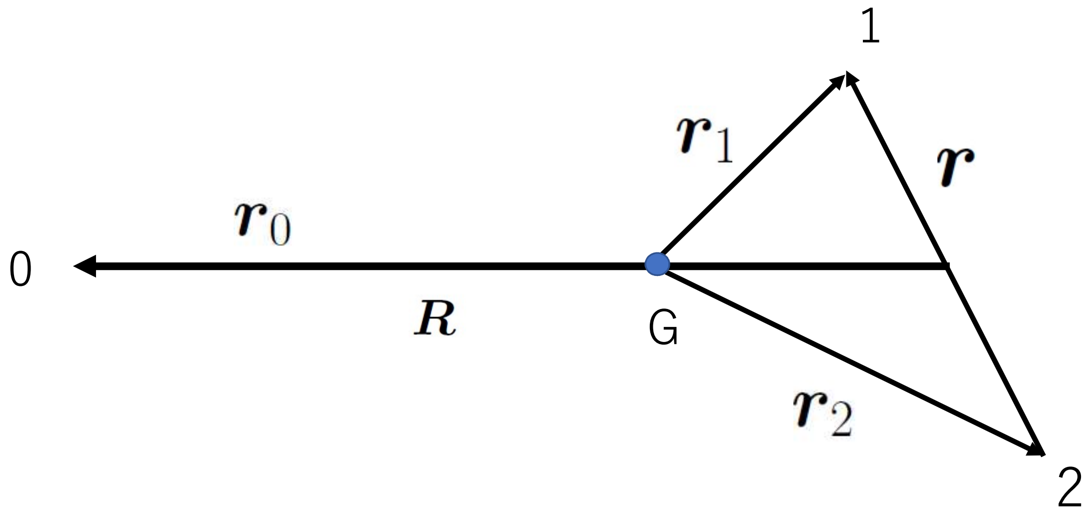
$$d(p, n)pp$$

Simplifications

- Ignore spins, isospins, Pauli principle,
- $m_p = m_n = m_N$
- Consider only the inclusive cross section

1.1. Formalism

Use the coordinate system



$$\boldsymbol{r}_0 + \boldsymbol{r}_1 + \boldsymbol{r}_2 = 0$$
$$\boldsymbol{r}_0 - \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2} = \boldsymbol{R}$$
$$\boldsymbol{r}_1 - \boldsymbol{r}_2 = \boldsymbol{r}$$

Final state momenta $(\boldsymbol{p}_0,\ \boldsymbol{p}_1,\ \boldsymbol{p}_2)$

$$\boldsymbol{p}_0 = \boldsymbol{k}_f$$
$$\boldsymbol{P}_{\rm res} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = -\boldsymbol{k}_f,$$
$$\frac{\boldsymbol{p}_1 - \boldsymbol{p}_2}{2} = \boldsymbol{\kappa}$$

- Initial state

$$|i\rangle = \phi_d(\boldsymbol{r})\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{R}}$$

- Final state

$$|f\rangle = \mathrm{e}^{\mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{R}}\phi_{pp}(\boldsymbol{\kappa};\boldsymbol{r})$$

Asymptotic form

$$\phi_{pp}(\boldsymbol{\kappa};\boldsymbol{r}) \sim \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{r}}$$

ϕ_d : deuteron wave function

ϕ_{pp}: wave function of the final pp system

- Interaction

$$V(\boldsymbol{r}_0 - \boldsymbol{r}_1) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\tilde{V}_{pn}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}_0-\boldsymbol{r}_1)}$$
$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\tilde{V}_{pn}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{R}-\frac{1}{2}\boldsymbol{r})}$$

- T-matrix

$$
\begin{aligned}
T_{fi} &= \langle f|V|i\rangle \\
&= \tilde{V}(\boldsymbol{q}^*)\int d^3\boldsymbol{r}\,\phi^*_{pp}(\boldsymbol{\kappa};\boldsymbol{r})\phi_d(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\frac{\boldsymbol{q}^*}{2}\cdot\boldsymbol{r}} \\
&= \tilde{V}(\boldsymbol{q}^*)F_{pp,d}(\boldsymbol{\kappa};\frac{\boldsymbol{q}^*}{2})
\end{aligned}
$$

- Transition form factor

$$
F_{pp,d}(\boldsymbol{\kappa};\boldsymbol{q}) = \int d^3\boldsymbol{r}\,\phi^*_{pp}(\boldsymbol{\kappa};\boldsymbol{r})\phi_d(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}
$$

- Momentum transfer to the internal motion

$$
\boldsymbol{q} = \frac{\boldsymbol{q}^*}{2}
$$

[Comment]

On the center of mass problem

Why differs

$$\langle \phi_{pp} | \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \phi_d \rangle \quad \text{vs.} \quad \langle \phi_{pp} | \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^*\cdot\boldsymbol{r}_1} | \phi_d \rangle$$

We must take the replacement

$$\boldsymbol{r}_1 \longrightarrow \boldsymbol{r}_1 - \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2} = \frac{\boldsymbol{r}}{2}$$

then we get

$$\mathrm{e}^{-\mathrm{i}\boldsymbol{q}^*\cdot\boldsymbol{r}_1} \longrightarrow \mathrm{e}^{-\mathrm{i}\boldsymbol{q}^*\cdot\frac{\boldsymbol{r}}{2}} = \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}, \quad (\boldsymbol{q} = \frac{\boldsymbol{q}^*}{2})$$

● Inclusive cross section

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K|\tilde{V}(\boldsymbol{q}^*)|^2 \times \int \frac{d^3\boldsymbol{\kappa}}{(2\pi)^3} |F_{pp,d}(\boldsymbol{\kappa}; \frac{\boldsymbol{q}^*}{2})|^2 \delta(\omega^* - \bar{\omega}^*)$$

where

$$\bar{\omega}^* = \frac{m_p^2 - m_d^2 - m_n^2 + M_{pp}^2}{2\sqrt{s}}$$

with the invariant mass of the *pp* system

$$M_{pp}^2 = (E_1^* + E_2^*)^2 - k_f^2$$

- Excitation Energy

Assume

Invariant mass = Mass + Internal energy

we may write

$$M_{pp} = 2m_p + \frac{\kappa^2}{m_p} = m_d + E_x$$

E_x : Excitation energy of the 2N system

(with respect to the target ground state)

Then we get

$$\bar{\omega}^* = \frac{m_p^2 - m_d^2 - m_n^2 + m_d^2 + 2m_d E_x + E_x^2}{2\sqrt{s}}$$

$$\approx \frac{m_d}{\sqrt{s}} E_x$$

Introduce

$$\omega = \frac{\sqrt{s}}{m_d}\omega^*$$

which means

Energy transfer to the internal motion

● Double differential cross section

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K\frac{\sqrt{s}}{m_d}|\tilde{V}_(\boldsymbol{q}^*)|^2$$
$$\times \int \frac{d^3\boldsymbol{\kappa}}{(2\pi)^3}|F_{pp,d}(\boldsymbol{\kappa};\boldsymbol{q})|^2\delta(\omega - E_x)$$

with

$$E_x = 2m_p - m_d + \frac{\kappa^2}{m_p}$$

With spins, isospins, antisymmetrization, etc., the formula is more complicate.

See A. Itabashi, K. Aizawa, and M. Ichimura, Prog. Theoret. Phys, **91** 91(1994)

1.2. Practical calculation

(1) Fix ω^*, then $\Longrightarrow \omega \Longrightarrow E_x \Longrightarrow \bar{\kappa}$

(2) Solve the Schrödinger equations

- for the deuteron

$$H_{2N}\phi_d(\boldsymbol{r}) = (2m_N - \epsilon_d)\phi_d(\boldsymbol{r})$$

- for the pp system

$$H_{2N}\phi^*_{pp}(\bar{\boldsymbol{\kappa}};\boldsymbol{r}) = (2m_N + \frac{\kappa^2}{m_p})\phi^*_{pp}(\boldsymbol{\kappa};\boldsymbol{r})$$

for low partial waves($\ell \leq 2$).
Use the plane wave for higher partial waves

(3) Calculate the form factor

$$F_{pp,d}(\boldsymbol{\kappa};\boldsymbol{q}) = \int d^3\boldsymbol{r}\phi^*_{pp}(\boldsymbol{\kappa};\boldsymbol{r})\phi_d(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}$$

by the partial wave expansion.

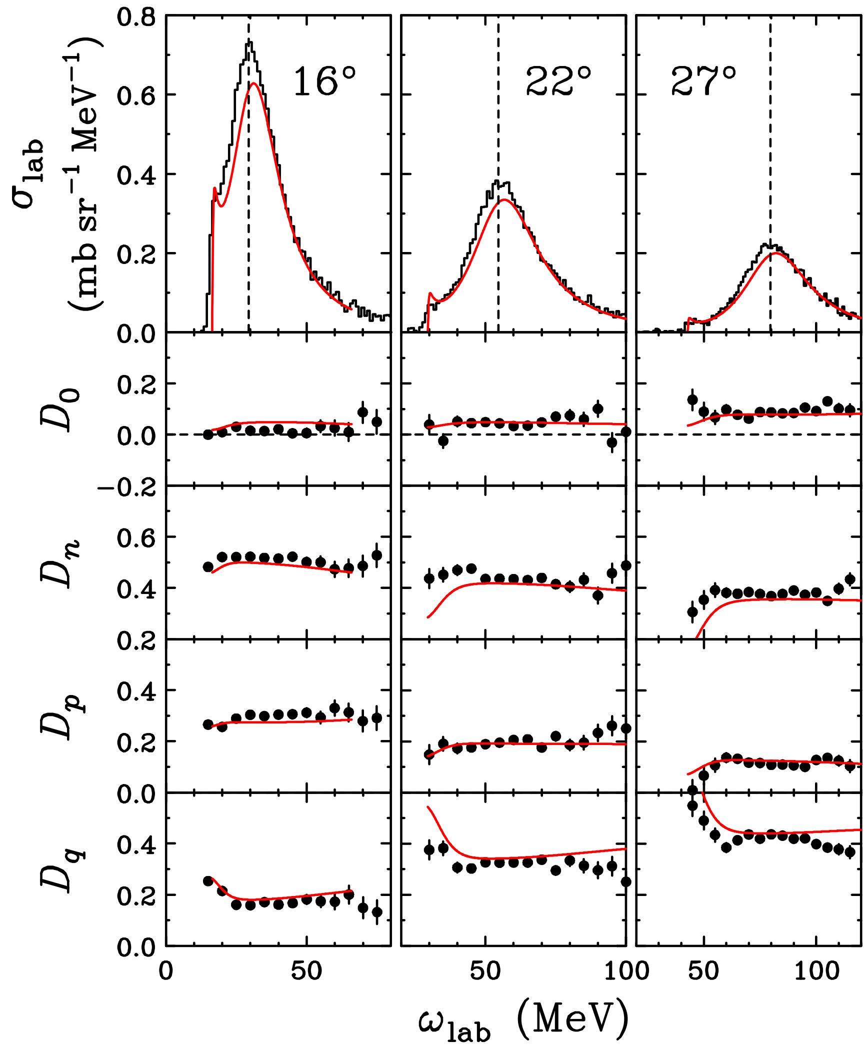
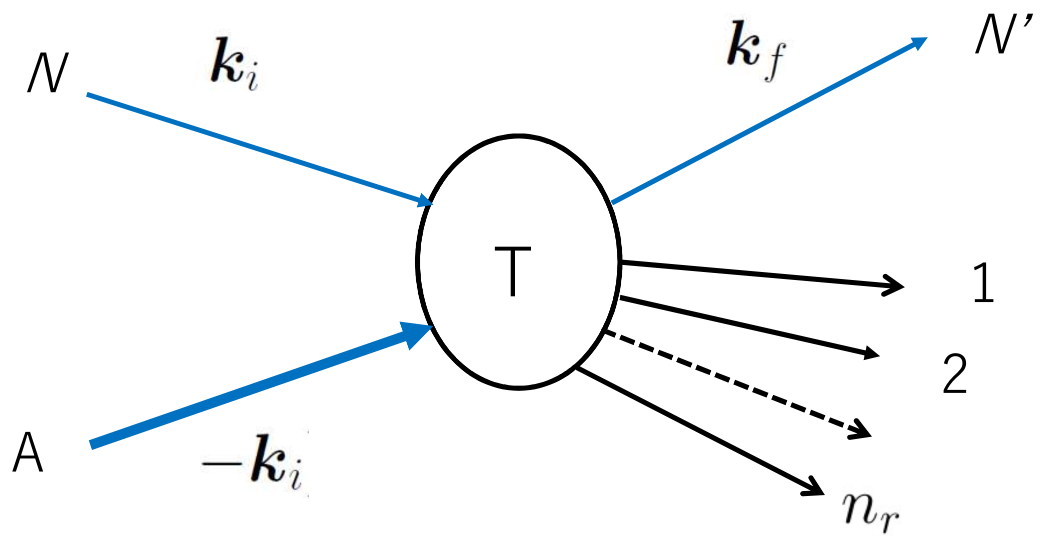


図 1　Cross sections and polarization observables D_i of the ^{2}H(p,n) reaction at $T_p = 345$ MeV. T. Wakasa et al., Phys. Rev. C69 (2004) 044602

2. Response function formalism

The above method works for very limited cases, such as a few nucleon target d,^{3}He.

Let's consider a method applicable for more general cases.



$$A \ + \ N \ \longrightarrow \ N' \ + \ X \text{ (anything)}$$

n_r : number of outgoing clusters
in the residual system X.

Take only the direct term
(treat the exchange term
by the pseudo-potential.)

T-matrix is given by

$$T_{fi} = \tilde{V}(\boldsymbol{q}^*) \ \langle \Phi_X | \tilde{\rho}(\boldsymbol{q}^*) | \Phi_A \rangle$$

with the transition density

$$\tilde{\rho}((\boldsymbol{p}) = \sum_{k=1}^{A} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}_k}$$

X represents

$$X = (n_r, \boldsymbol{p}_1, \cdots \boldsymbol{p}_{n_r}, \alpha)$$

and $f = (\boldsymbol{k}', X)$

α : the quantum number other than

$(n_r, \boldsymbol{p}_1, \cdots \boldsymbol{p}_{n_r})$

Use the notation Σ_X, which means

$$\sum_X = \sum_{n_r,\alpha} \int \frac{d^3\boldsymbol{p}_1^*}{(2\pi)^3} \cdots \frac{d^3\boldsymbol{p}_{n_r}^*}{(2\pi)^3} (2\pi)^3 \delta \left(\boldsymbol{k}' + \sum_{k=1}^{n_r} \boldsymbol{p}_k^* \right)$$

Inclusive double differential cross section is now given by

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \sum_X |T_{fi}|^2 \, \delta(\omega^* - \bar{\omega}^*)$$

where

$$\bar{\omega}^* = \frac{m_N^2 - m_A^2 - m_{N'}^2 + M_X^2}{2\sqrt{s}}$$

M_X : invariant mass of the system X.

Write

$$M_X = m_A + E_x^X$$

E_x^X : Excitation energy of the system X with respect to the target ground state.

$$\bar{\omega}^* \approx \frac{m_A}{\sqrt{s}} E_x^X$$

We get

Double Differential Cross Section

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \frac{\sqrt{s}}{m_A} \sum_X |T_{fi}|^2 \, \delta(\omega - E_x^X)$$

with the energy transfer to the internal motion of X

$$\omega = \frac{m_A}{\sqrt{s}} \, \omega^*$$

Rewrite this as

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \frac{\sqrt{s}}{m_A} |\tilde{V}(\boldsymbol{q}^*)|^2 R_\rho(\boldsymbol{q}^*)$$

with **Response function** for ρ

$$R_\rho(\omega, \boldsymbol{q}^*) \equiv \sum_X |\langle \Phi_X | \tilde{\rho}(\boldsymbol{q}^*) | \Phi_A \rangle|^2 \delta(\omega - E_x^X)$$

- ○ The structure part $R_\rho(\omega, \boldsymbol{q}^*)$ is well separated !
- ○ The question is how to calculate the infinite sum Σ_X.

  A main theme !!!

Introduce

Hamiltonian of the system $A(= X)$

$$H_A \Phi_X = E_x^X \Phi_X, \quad (E_x^X = 0, \text{ if } X = A)$$

We can express the response function as

$$\begin{aligned}
& R_\rho(\omega, \boldsymbol{q}^*) \\
= & \sum_X |\langle \Phi_X | \tilde{\rho}(\boldsymbol{q}^*) | \Phi_A \rangle|^2 \delta(\omega - E_x^X) \\
= & -\frac{1}{\pi} \mathrm{Im} \left[\sum_X \langle \Phi_A | \tilde{\rho}^\dagger(\boldsymbol{q}^*) | \Phi_X \rangle \frac{1}{\omega - E_x^X + \mathrm{i}\eta} \right. \\
\times & \qquad \left. \langle \Phi_X | \tilde{\rho}(\boldsymbol{q}^*) | \Phi_A \rangle \right] \\
= & -\frac{1}{\pi} \mathrm{Im} \left[\langle \Phi_A | \tilde{\rho}^\dagger(\boldsymbol{q}^*) \frac{1}{\omega - H_A + \mathrm{i}\eta} \tilde{\rho}(\boldsymbol{q}^*) | \Phi_A \rangle \right]
\end{aligned}$$

Σ_X and Φ_X disappeared !

Can we calculate this response function ?

3. Summary

- Inclusive double differential cross section

$$\frac{d^2\sigma}{d\omega^* d\Omega} = K \frac{\sqrt{s}}{m_A} |\tilde{V}(\boldsymbol{q}^*)|^2 \, R_\rho(\omega, \boldsymbol{q}^*)$$

- Response function

$$R_\rho(\omega, \boldsymbol{q}) = -\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|\tilde{\rho}^\dagger(\boldsymbol{q})\frac{1}{\omega - H_A + \mathrm{i}\eta}\tilde{\rho}(\boldsymbol{q})|\Phi_A\rangle$$

- Key points

(1) Factorization of the reaction part and the structure part.

(2) Each part depends only on the momentum transfer $\boldsymbol{q}^*$ as to the spatial degree of freedom.

(3) The feature (2) is due to the fact that the interaction V is a local operator i.e $V = V(\boldsymbol{r}_0 - \boldsymbol{r}_k)$.

(4) To get reliable structure information, we must know the reliable reaction part, especially the interaction V.

(5) Infinite sum Σ_X is replaced by the expectation value of the target state, i.e. Response function of the one body operator.

(6) In a certain approximation such as HF, TDA, RPA, etc. these response functions are calculable as will be discussed.

[Caution]

Distinguish the three energy transfers

ω^{lab} in the lab frame

ω^{*} in the cm frame

ω to the internal state

= excitation energy

of the residual system, E_x,

with respect to the target ground state